

SUBSTITUTIONS AND $\frac{1}{2}$ -DISCREPANCY OF $\{n\theta + x\}$

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ABSTRACT. The sequence of $1/2$ -discrepancy sums of $\{x + i\theta \bmod 1\}$ is realized through a sequence of substitutions on an alphabet of three symbols; particular attention is paid to $x = 0$. The first application is to show that any asymptotic growth rate of the discrepancy sums not trivially forbidden may be achieved. A second application is to show that for badly approximable θ and any x the range of values taken over $i = 0, 1, \dots, n-1$ is asymptotically similar to $\log(n)$, a stronger conclusion than given by the Denjoy-Koksma inequality.

1. INTRODUCTION

Given an irrational θ and some $x \in [0, 1) = S^1$ (all addition in S^1 is taken modulo one), let

$$(1) \quad f(x) = \chi_{[0, 1/2)}(x) - \chi_{[1/2, 1)}(x).$$

With θ fixed, the $1/2$ -discrepancy sums of the sequence $\{x + i\theta\}$ are given by

$$S_n(x) = \sum_{i=0}^{n-1} f(x + i\theta).$$

Two results are classical in this setting, for any irrational θ and for all x :

$$(2) \quad S_n(x) \in o(n), \quad S_n(x) \notin O(1).$$

The first restriction is due to unique ergodicity of the underlying rotation, and the second is a theorem of Kesten [5].

We will use standard continued fraction notation; partial quotients are denoted $a_i(\theta)$, and convergents are denoted $p_i(\theta)/q_i(\theta)$. When θ is clear from context we will simply write a_i , p_i and q_i . The distance from x to the nearest integer is denoted $\|x\|$. As $\theta \in (0, 1)$ without loss of generality, we will assume that $a_0(\theta) = 0$ and omit this term, writing simply

$$\theta = [a_1, a_2, a_3, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}}.$$

All necessary background in continued fractions may be found in [6]. The *Gauss map* will be denoted by γ , and acts as the non-invertible shift on the sequence of

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partial quotients:

$$(3) \quad \gamma(\theta) = \frac{1}{\theta} \bmod 1, \quad \gamma([a_1, a_2, \dots]) = [a_2, a_3, \dots].$$

Our goal is to investigate what behavior is possible for the sequence $S_n(x)$ within the constraints of (2). Because the sequence S_n is not monotone, however, it will be more convenient to consider the following sequences, which track the *maximal* and *minimal* discrepancies, as well as the *range* of values taken:

$$(4) \quad M_n(x) = \max\{S_i(x) : i = 1, \dots, n-1\},$$

$$(5) \quad m_n(x) = \min\{S_i(x) : i = 1, \dots, n-1\},$$

$$(6) \quad \rho_n(x) = M_n(x) - m_n(x) + 1.$$

It is worth clarifying that m_n is taken as a minimum over *integers*, and as such can generally be expected to be negative. It is a matter of later convenience that $i = 0$ is not considered: for example, $M_1(0) = m_1(0) = S_1(0) = 1$.

We will develop a renormalization procedure through which the sequence of values $f(x + i\theta)$ can be determined from a sequence of substitutions. Let $\theta < 1/2$ and $A = [0, 1/2)$, $B = [1/2, 1 - \theta)$, $C = [1 - \theta, 1)$. If we wish to change which interval certain endpoints belong to (for example, if we wish for A to be closed and B to be open), we will say that we make a *change of endpoints* of the intervals A , B , and C . Our central result is the following:

Theorem 1.1. *Given any irrational θ and any $x \in [0, 1)$, there is a sequence of words ω_i (some of which may be empty) and substitutions σ_i (infinitely many are not identity) both defined on the alphabet $\{A, B, C\}$, given by a dynamic process depending on x and θ , such that the infinite word given by*

$$(7) \quad \omega_0 \sigma_0 (\omega_1 \sigma_1 (\omega_2 \sigma_2 (\dots)))$$

encodes the orbit of x up to at most two errors. Alternately, the coding is exact up to a change of endpoints of the intervals A , B and C . The dependence of σ_i on θ and ω_i on (x, θ) is explicit.

There is one special point $x(\theta)$ for which all ω_i may be taken to be the empty word, in which case the infinite word

$$(8) \quad \lim_{n \rightarrow \infty} (\sigma_0 \circ \sigma_1 \circ \dots \circ \sigma_{n-1})(\omega)$$

will encode the orbit of $x(\theta)$ regardless of the choice of nonempty word ω . The orbit of zero can alternately be determined by

$$(9) \quad \lim_{n \rightarrow \infty} (\sigma'_0 \circ \sigma'_1 \circ \dots \circ \sigma'_{n-1})(\omega'_{n-1}),$$

where σ'_n are either substitutions or a different map. This distinction and the word ω'_n are explicitly presented.

We will include some remarks regarding the point $x(\theta)$ (including a complete characterization of those θ for which $x(\theta) = 0$ in Proposition 4.3), as well as proving that the sequence of substitutions σ_i is eventually periodic if and only if θ is a quadratic surd (Proposition 4.4).

As $(0, 1/2) \subset A$ and $(1/2, 1) \subset (B \cup C)$, any change of endpoints is completely irrelevant to the asymptotic growth rates of $M_n(x)$, $m_n(x)$, and $\rho_n(x)$. While Theorem 1.1 provides a way to produce the orbit of an arbitrary point, computation of the words ω_i is a nontrivial task. However, for the special point $x(\theta)$ and for 0,

the process is much simpler. We will show that given any growth condition that does not violate (2), such behavior is seen to be possible:

Theorem 1.2. *Suppose that $\{c_n\}$ and $\{d_n\}$ are two increasing sequences of positive real numbers, both in $o(n)$, the differences*

$$\Delta c_n = c_{n+1} - c_n$$

are in $O(1)$ (similarly for $\{\Delta d_n\}$), and at least one of $\{c_n\}$, $\{d_n\}$ is divergent. Then there is a dense set of θ such that if $\{c_n\}$ is divergent, then

$$\limsup_{n \rightarrow \infty} \frac{M_n(0)}{c_n} = 1,$$

while if $\{c_n\}$ is bounded then so is $M_n(0)$. Similarly, if $\{d_n\}$ is divergent, then

$$\limsup_{n \rightarrow \infty} \frac{|m_n(0)|}{d_n} = 1,$$

while if $\{d_n\}$ is bounded then so is $m_n(0)$.

A closely related result concerns the sequence of values $M_n(x)/|m_n(x)|$:

Theorem 1.3. *Let $0 \leq r_1 \leq r_2 \leq \infty$. Then there is a dense set of θ such that the set of accumulation points of the sequence*

$$\left\{ \frac{M_n(0)}{|m_n(0)|} : n = 0, 1, 2, \dots \right\}$$

is the interval $[r_1, r_2]$.

We will also include a partial rederivation of [2, Theorem 1] in Corollary 5.3: a characterization of those θ for which $S_n(\theta) \geq 0$ for all $n \geq 0$.

A classical application of the Denjoy-Koksma inequality is that if the $a_i(\theta)$ are drawn from a finite set (such θ are said to be *badly approximable* or *of finite type*), then $S_n(x) \in O(\log n)$.

Theorem 1.4. *If θ is of finite type, then for all x we have $\rho_n(x) \sim \log n$, meaning that the ratio is bounded away from both zero and infinity.*

Corollary 1.5. *If θ is of finite type, then $|S_n(x)| \notin o(\log n)$ for every x , and*

$$m_n(x) \in o(\log n) \implies M_n(x) \sim \log n,$$

and vice-versa.

If $A \cup B$ represents a single interval, then as S^1 has been partitioned into two intervals of length θ and $1 - \theta$, the analogous problem would be to encode the *Sturmian sequences*, and generating Sturmian sequences using a sequence of substitutions is intimately related to continued fraction expansions for numbers: see for example [3, Chapter 6]. The study of substitutions as they relate to discrepancy sequences of different intervals has been initiated before [1], in this paper our approach is different:

- the interval $[0, 1/2]$ is not *dynamically defined*, i.e. not dependent on θ (although it is fixed),
- we develop an approach for all θ (not just quadratic surds, though the process is nicest in this setting),
- we generate the orbit of any starting point x (though $x = 0$ is one particularly nice case that we investigate).

2. SYMBOL SPACES, ENCODINGS, AND SUBSTITUTIONS

All background material pertaining to common definitions in symbolic dynamics and substitution systems may be found in [3, Chapter 1]; we present here only a short summary of specific notation used herein. Let $\mathcal{A} = \{A, B, C\}$, and denote by \mathcal{A}^* the *free monoid* on \mathcal{A} . Given $\omega \in \mathcal{A}^*$, we denote

$$\omega = (\omega)_0(\omega)_1 \dots (\omega)_{n-1},$$

and say that ω is a *word of length n* with *letters* $(\omega)_i$ drawn from the *alphabet* \mathcal{A} . Note that ω_i will refer to a sequence of words indexed by i , while $(\omega)_i$ will denote the individual letters of a fixed word ω . This similarity is a potential source of confusion, but the latter notation is much more common in this work: we will rarely refer to specific letters in a given word.

Denote by $|\omega|$ the length of ω . Elements in \mathcal{A}^* multiply by concatenation, and we adopt power notation for this operation: $(AB)^3 = ABABAB$, for example. The empty word (the identity under concatenation) we denote \emptyset . A *factor* of ω (of finite or infinite length) is some finite word ψ of length n such that there is some i for which

$$(\psi)_j = (\omega)_{i+j}, \quad j = 0, 1, \dots, n-1.$$

If $i = 0$ then we say ψ is an *left factor* of ω , and we say ψ is a *right factor* of ω if $(\psi)_{n-1} = (\omega)_{|\omega|-1}$. The factor ψ will be called *proper* if $\psi \notin \{\omega, \emptyset\}$.

Any map $\sigma : \mathcal{A} \rightarrow \mathcal{A}^*$ may be extended to a map on \mathcal{A}^* by requiring it to be a homomorphism. The following is nonstandard but natural. Endow $\mathcal{A}^{\mathbb{N}}$ with the cylinder topology, and let a finite word $\omega \in \mathcal{A}^*$ represent a clopen set: the set of all elements of $\mathcal{A}^{\mathbb{N}}$ with left factor ω . We may then further extend σ to a map on $\mathcal{A}^{\mathbb{N}}$ by defining

$$\sigma(\omega) = \bigcap_{i=0}^{\infty} \sigma((\omega)_0(\omega)_1 \dots (\omega)_{i-1}).$$

In all of these situations we refer to σ as a *substitution*.

Given a sequence of words $\omega_0, \omega_1, \dots$ such that ω_i is a left factor of ω_{i+1} , if

$$\bigcap_{i=0}^{\infty} \omega_i = \{x\},$$

then we say that $x \in \mathcal{A}^{\mathbb{N}}$ is the *limit* of the words ω_i .

Now consider the space $S^1 = [0, 1)$ with the map $R_\theta(x) = x + \theta \pmod{1}$ for some irrational θ . Suppose that X is partitioned into three intervals A , B , and C . Then given a word ω , we say that ω *encodes the orbit of x* if for all $i \leq |\omega| - 1$ we have

$$(\omega)_i = A \iff x + i\theta \in A,$$

and similarly for B and C . Given a partition, then, to each $x \in S^1$ we may identify an infinite word $\omega \in \Omega$: the infinite word which encodes the (forward) orbit of x .

Let \mathcal{D} be the discontinuities of $(f \circ R_\theta^i)(x)$ for $i = 0, 1, 2, \dots$:

$$\mathcal{D} = \{-i\theta, -i\theta + 1/2\}, \quad i = 0, 1, 2, \dots$$

For each $x \in \mathcal{D}$, then, we replace $x \in S^1$ with two points, a right and left limit, denoted x^+ and x^- . We set

$$R_\theta(0^+) = R_\theta(1^-) = \theta,$$

and similarly for $(1/2)^\pm$; while this makes the rotation two-to-one at these points, note that with respect to the alphabet \mathcal{A} , the symbolic coding for the forward orbit of θ^+ and θ^- are identical, so we do not distinguish them. We still denote our space by S^1 . We may now make each of A , B and C closed, although we have made S^1 totally disconnected.

Given an irrational θ , partition $S^1 = [0^+, 1^-]$ according to Table 1 and in a slight abuse of notation let S^1 be the set of all words which encode orbits with respect to these conventions.

$\theta < 1/2$	$\theta > 1/2$
$A = [0^+, \frac{1}{2}^-]$	$C = [0^+, (1 - \theta)^-]$
$B = [\frac{1}{2}^+, (1 - \theta)^-]$	$B = [(1 - \theta)^+, \frac{1}{2}^-]$
$C = [(1 - \theta)^+, 1]$	$A = [\frac{1}{2}^+, 1^-]$

TABLE 1. The partition $S^1 = A \cup B \cup C$ depending on θ .

The following lemma is immediate, and immediately explains the apparent ambiguity in the statement of Theorem 1.1:

Lemma 2.1. *If ω is an infinite word encoding the orbit of a point $x \in S^1$ under rotation by θ , then ω encodes the orbit of some $x \in S^1$ without the introduction of \mathcal{D} with at most two errors. Alternately the coding is exact up a change of endpoints of the intervals A , B and C .*

Proof. The orbit of any point can hit the endpoints of A , B and C at most twice. \square

3. THE RENORMALIZATION PROCEDURE

Recall γ , the Gauss map (3); we define a similar map.

$$(10) \quad g([a_1, a_2, a_3, \dots]) = \begin{cases} [a_3, a_4, \dots] = \gamma^2(\theta) & (a_1 = 0 \bmod 2) \\ [1, a_2, a_3, \dots] = \frac{1}{1+\gamma(\theta)} & (a_1 = 1 \bmod 2, a_1 \neq 1) \\ [a_2 + 1, a_3, \dots] = 1 - \theta & (a_1 = 1). \end{cases}$$

Note that if $\theta > 1/2$, then necessarily $g(\theta) < 1/2$. It will be convenient to define

$$(11) \quad E(x) = \max\{n \leq x : n \in \mathbb{Z}, n = 0 \bmod 2\}.$$

The triplet $\{X, \mu, T\}$ refers to a compact probability space $\{X, \mu\}$ and a continuous transformation T on X which preserves μ . Given irrational θ , we denote

$$(12) \quad \theta_n = g^n(\theta), \quad \delta_n = 1 - E(a_1(\theta_n))\theta_n, \quad I_n = \{S^1, \mu, R_{\theta_n}\}.$$

Note that $\delta_n = 1$ if and only if $\theta > 1/2$; otherwise $\delta_n < 1/2$.

Partition each I_n into intervals A , B and C according to Table 1, and recall that by convention we have disconnected each I_n such that all iterates of the characteristic functions of A , B and C under $R_{\theta_n}^i$ are continuous. Given $\{X, \mu, T\}$ and a set $S \subset X$, the *return time to S* is given by

$$n(x) = \min\{n > 0 : T^n(x) \in S\}.$$

As irrational rotations are minimal, $n(x)$ will be defined for all $x \in S^1$ if S is an interval of positive length. The *induced system* on S is defined by

$$\{S, \mu|_S, T|_S\},$$

where $T|_S(x) = T^{n(x)}(x)$ for all $x \in S$. Define $I'_{n+1} \subset I_n$ by

$$I'_{n+1} = [0^+, \delta_n^-].$$

Finally, define the substitutions $\sigma_n = \sigma(\theta_n)$ according to Table 2, and define the functions $\varphi_n = \varphi(\theta_n)$ according to:

$$(13) \quad \varphi(x) = \begin{cases} 1 - x & (a_1(\theta) = 1) \\ \delta_n^{-1}x & (a_1(\theta) \neq 1) \end{cases}$$

Case	Substitution
$a_1 = 2k, a_3 \neq 1$	$A \rightarrow (A^{k+1}B^{k-1}C)(A^k B^{k-1}C)^{a_2-1}$ $B \rightarrow (A^k B^k C)(A^k B^{k-1}C)^{a_2-1}$ $C \rightarrow (A^k B^k C)(A^k B^{k-1}C)^{a_2}$
$a_1 = 2k, a_3 = 1$	$A \rightarrow (A^k B^k C)(A^k B^{k-1}C)^{a_2}$ $B \rightarrow (A^{k+1}B^{k-1}C)(A^k B^{k-1}C)^{a_2}$ $C \rightarrow (A^{k+1}B^{k-1}C)(A^k B^{k-1}C)^{a_2-1}$
$a_1 = 2k + 1$	$A \rightarrow A^k B^k C$ $B \rightarrow A^{k+1} B^{k-1} C$ $C \rightarrow A$
$a_1 = 1$	$A \rightarrow A$ $B \rightarrow B$ $C \rightarrow C$

TABLE 2. The substitution σ as a function of θ .

Lemma 3.1. *Suppose that $\theta < 1/2$, $E(a_1(\theta)) = 2k$, and*

$$(1 - 2k\theta)^+ \leq x \leq \left(\frac{1}{2} - (k-1)\theta\right)^-.$$

Then the orbit of x begins $A^k B^{k-1} C$.

Proof. The assumption $\theta < 1/2$ tells us how to partition S^1 according to Table 1 as well as guaranteeing that $k \geq 1$. Note that the lower inequality certainly guarantees that

$$\frac{1}{2} - k\theta < x \leq \left(\frac{1}{2} - (k-1)\theta\right)^-,$$

which tells us that $x + i\theta \leq (1/2)^-$ for $i = 0, 1, \dots, (k-1)$, while $x + k\theta > 1/2$. So the coding of the orbit of x begins with exactly A^k before seeing either B or C . As we know

$$(1 - 2k\theta)^+ \leq x < 1 - (2k-1)\theta,$$

we know that we have $x + (2k-1)\theta < 1$, while $x + 2k\theta \geq 1^+$. Therefore, once we have accounted for the points $x + i\theta$ for $i = 0, 1, \dots, k-1$, the terms $i = k, k+1, \dots, (2k-1)$ must all belong to either B or C . That C is an interval of length exactly θ guarantees that exactly the final term is C . The rest of the terms (if there are any) are therefore B . \square

Proposition 3.2. *We have the measurable and continuous isomorphism*

$$\left\{ I'_{n+1}, \mu|_{I'_{n+1}}, (R_{\theta_n})|_{I_{n+1}} \right\} \xrightarrow{\varphi_n} \left\{ I_{n+1}, \mu, R_{\theta_{n+1}} \right\}.$$

Furthermore, for all $x \in A \subset I_{n+1}$, the word $\sigma_n(A)$ encodes the orbit of $\varphi^{-1}(x)$ through its return to I'_{n+1} (the encoding is with respect to the partition A, B, C in I_n), and similarly for B and C .

Proof. In the case that $\theta_n > 1/2$, then $\theta_{n+1} = 1 - \theta_n$ and $I'_{n+1} = [0^+, 1^-]$. However, by referring to Table 1, we see that the intervals A, B and C exactly reflect the reversal of orientation given by $\varphi_n(x) = 1 - x$, and the substitution σ_n is identity. So we proceed on the assumption that $\theta_n < 1/2$: in I_n we have

$$A = [0^+, 1/2^-], \quad B = [1/2^+, (1 - \theta)^-], \quad C = [(1 - \theta)^+, 1^-].$$

Then φ_n is scalar multiplication by δ_n^{-1} , so there are only two things to show:

- The first-return map $(R_{\theta_n})|_{I'_{n+1}}$ is rotation by θ_{n+1} , after rescaling by φ_n , and
- the substitution σ_n encodes the correct information.

There are three cases to consider: $a_1(\theta_n) = 1 \bmod 2$, or $a_1(\theta_n) = 0 \bmod 2$ with the sub-cases $a_3(\theta_n) = 1$ or $\neq 1$. Assume for now that $a_1(\theta_n) = 0 \bmod 2$ and $a_3(\theta_n) = 1$.

As $a_1(\theta_n) = 0 \bmod 2$ and $a_3(\theta_n) = 1$, we have $g(\theta_n) = \gamma^2(\theta_n) > 1/2$, so in I_{n+1} we have

$$C = [0^+, (1 - \theta_{n+1})^-], \quad B = [(1 - \theta_{n+1})^+, 1/2^-], \quad A = [1/2^+, 1^-],$$

with corresponding preimages in I'_{n+1} scaled by δ_n . We will first verify that the intervals have the desired return times (which may be read from the length of the words $\sigma_n(A)$, $\sigma_n(B)$ and $\sigma_n(C)$) and that the induced map is indeed rotation by θ_{n+1} (up to scale δ_n). As $E(a_1(\theta_n)) = a_1(\theta_n)$ we have

$$\delta_n = \|q_1(\theta_n) \cdot \theta_n\|,$$

from which it follows that the return time of 0 is

$$n(0) = q_2 = a_1 a_2 + 1,$$

and one may now verify that the entire interval $\varphi_n^{-1}(C)$ has this return time; the preimage of the right endpoint of C under φ_n is exactly $1 - (q_1 + q_2)\theta_n$. The remaining points in I'_{n+1} have return time $q_2 + q_1$ and the induced map is a rotation by $q_2\theta_n$ on $[0^+, \delta_n^-]$; see Figure 1.

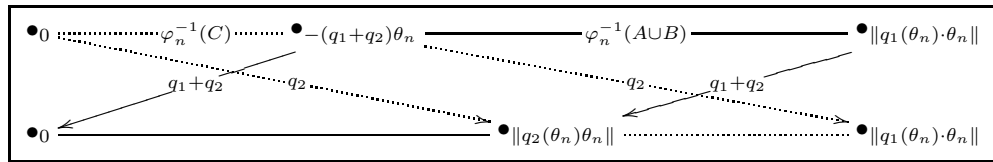


FIGURE 1. Return times for the case $a_1(\theta_n) = 0 \bmod 2$, $a_3(\theta_n) = 1$.

At this point we may verify that the rotation is by $g(\theta_n)$, up to scale:

$$\begin{aligned}
\frac{\|q_2(\theta_n) \cdot \theta_n\|}{\delta_n} &= \frac{q_2(\theta_n) \cdot \theta_n - p_2(\theta_n)}{1 - q_1(\theta_n) \cdot \theta_n} \\
&= \frac{(a_1 a_2 + 1)\theta_n - a_2}{1 - a_1 \theta_n} \\
&= \frac{a_2 \left(a_1 - \frac{1}{\theta_n}\right) + 1}{\frac{1}{\theta_n} - a_1} \\
&= \frac{1 - a_2 \gamma(\theta_n)}{\gamma(\theta_n)} \\
&= \gamma^2(\theta_n).
\end{aligned}$$

Now suppose that $x \in \varphi^{-1}(B)$, and for convenience denote $E(a_1) = a_1 = 2k$. Clearly, the orbit of x begins with a point in A (in I_n , as $A = [0^+, 1/2^-]$ contains $[0^+, \delta_n^-]$). As $x < 1/2 - k\theta_n$, however, we have

$$(1 - 2k\theta_n)^+ \leq x + \theta_n \leq ((1/2) - (k-1)\theta_n)^-,$$

so by Lemma 3.1, we may concatenate the word $A^k B^{k-1} C$ to this initial A . Since $2k = a_1$, we now have

$$x + \theta_n + (2k\theta_n) < x + \theta_n \leq ((1/2) - (k-1)\theta_n)^-.$$

Either we have returned to I'_{n+1} , in which case we are done, or we have not, in which case we apply Lemma 3.1 again, repeating until we return to I'_{n+1} , which must take a total of $q_2 + q_1 = a_1(a_2 + 1) + 1$ steps.

For those points in the interval $\varphi_n^{-1}(a)$, note that the only discontinuity of $R_{\theta_n}^i$ for $i = 0, 1, \dots, q_2$ to distinguish the orbits compared to points in $\varphi_n^{-1}(A)$ is the point $1/2 - k\theta$, which will change the single term $x + k\theta$ from an 'A' to a 'B'. Points in $\varphi_n^{-1}(C)$ are considered identically to those in $\varphi_n^{-1}(B)$, noting that the shorter return time requires one fewer concatenation of $A^{a_1} B^{a_1-1} C$.

The other cases are similarly considered; the case $a_1(\theta_n) = 0 \bmod 2$, $a_3(\theta_n) \neq 1$ is nearly identical, while for the case $a_1(\theta_n) = 1 \bmod 1$, $\neq 1$ we have $\delta_n > \theta_n$, so the return time of 0^+ is one, explaining the much shorter substitution $\sigma_n(C) = A$ in this case. \square

Denote the iterated pull-back of I_n into I_0 by

$$(14) \quad \tilde{I}_n = (\varphi_0^{-1} \circ \dots \circ \varphi_{n-1}^{-1})(I_n).$$

Corollary 3.3. *We have the measurable and continuous isomorphism*

$$\left\{ \tilde{I}_n, \mu|_{\tilde{I}_n}, (R_\theta) |_{\tilde{I}_n} \right\} \xrightarrow{(\varphi_{n-1} \circ \dots \circ \varphi_0)} \{I_n, \mu, R_{\theta_n}\}.$$

Furthermore, for any $x \in A \subset I_n$, the word $(\sigma_0 \circ \dots \circ \sigma_{n-1})(A)$ encodes the orbit of $(\varphi_0^{-1} \circ \dots \circ \varphi_{n-1}^{-1})(x)$ in I_0 through its return to \tilde{I}_n , and similarly for B, C .

4. PROOF OF THEOREM 1.1

The proof of (8) is immediate in light of Corollary 3.3; the point $x(\theta)$ is given by

$$x(\theta) = \bigcap_{i=0}^{\infty} \tilde{I}_i,$$

where the \tilde{I}_i were defined in (14). This intersection is nonempty as the sets are nested closed intervals in the compact space S^1 . The length of \tilde{I}_n is given by

$$\delta_0 \cdot \delta_1 \cdots \delta_{n-1},$$

and we have already remarked that for $\theta_n < 1/2$, we have $\delta_n < 1/2$. As no two successive terms in the sequence $\theta_0, \theta_1, \dots$ may be larger than one half, the length tends to zero, and the intersection is either a singleton or a pair $\{x^-, x^+\}$. In the latter scenario, however, both x^- and x^+ would have identical coding of their forward orbits. As we did not ‘split’ the points $i\theta$ or $i\theta + 1/2$ for $i > 0$ when disconnecting S^1 , this is not possible.

As all non-identity substitutions map each letter to a word beginning in A , and all non-identity substitutions map A to a word of length at least three, and no two consecutive substitutions may be identity, it follows that the sequence of words

$$(\sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_{n-1})(\omega)$$

has a limit regardless of the choice of nonempty ω , and Corollary 3.3 shows that this word must encode the orbit of $x(\theta)$ in the disconnected version of S^1 . Lemma 2.1 finishes the proof of this portion of Theorem 1.1.

Let us now turn our attention to constructing the orbit of an arbitrary $x_0 \in S^1$. Define

$$x_1 = x_0 + i\theta, \quad i \in \{j \geq 0 : x + j\theta \in I'_1\},$$

and let ω_0 be the word which encodes the orbit of x_0 through its arrival to x_1 ; if $x_0 \in I'_1$, we may set ω_0 to be the empty word (though we are not required to do so). We now pass to the system I_1 , letting $(x_1 \in I_1) = \varphi_0(x_1 \in I'_1)$. We set x_2 to be a point in I'_2 which is in the orbit of x_1 , and let ω_1 be the word encoding this finite portion of the orbit, then pass to I_2 , etc. Equation (7) now follows from Proposition 3.2 so long as infinitely many $\omega_n \neq \emptyset$. We only have the option of letting all but finitely many ω_n be empty if x is a preimage of $x(\theta)$; we have already remarked in this case that the limiting word may be found handily.

A potential source of confusion at this point is the desire to claim that $x(\theta) = 0$, as we always construct $I'_{n+1} = [0^+, \delta_n^-]$. However, $\varphi_n(x) = 1 - x$ for those n such that $\theta_n > 1/2$. So $\varphi_n^{-1} \circ \varphi_{n+1}^{-1}$ pulls back I_{n+2} to the interval $[(1 - \delta_{n+1})^+, 1^-] \subset I_n$. Those θ for which $x(\theta) = 0$ will be addressed in Proposition 4.3.

Proposition 4.1. *Without loss of generality, ω_n may be required to either be empty, or a proper right factor of either $\sigma_n(A)$, $\sigma_n(B)$, or $\sigma_n(C)$.*

Proof. The images of $R_{\theta_n}^i(I'_{n+1})$ cover all of I_n through the return times, so any x may be viewed as returning to I'_{n+1} via a right factor of one of these words. If the return is through the entire word $\sigma_n(A)$, we would have begun with $x_n \in I'_{n+1}$ and could have set $\omega_n = \emptyset$. \square

Remark. One could alternately require that ω_n be nonempty by allowing all nonempty right factors of $\sigma_n(A)$, $\sigma_n(B)$, and $\sigma_n(C)$; instead of $\omega_n = \emptyset$ for $x \in I'_{n+1}$, let ω_n be σ applied to the letter encoding whichever interval in I_{n+1} contains $\varphi_n(x)$.

In order to construct the orbit of zero we will side-step this computation altogether:

Lemma 4.2. *Suppose that $\theta_n > 1/2$. Let Ω encode the orbit of 0^+ in the system I_n , and Υ encode the orbit of 0^+ in the system I_{n+1} . Then for all $i \geq 1$, $(\Omega)_i = (\Upsilon)_i$. For $i = 0$, $(\Omega)_0 = C$ while $(\Upsilon)_0 = A$.*

Proof. The isomorphism $\varphi_n(x) = 1 - x$ and the identity substitution σ_n ensures that Ω is identical to the coding of the orbit of 1^- in I_{n+1} . As the forward orbit of 0 under rotation by the irrational θ_n does not hit any other endpoints of the intervals A , B , and C , we have that the orbit of 1^- and 0^+ in the system I_{n+1} are identical after this initial term. \square

With this lemma in mind, then, define the map $\Psi(\omega)$ on both \mathcal{A}^* and $\mathcal{A}^{\mathbb{N}}$:

$$(15) \quad (\Psi\omega)_i = \begin{cases} C & (i = 0) \\ \omega_i & (i \neq 0). \end{cases}$$

Define the maps $\sigma'_n = \sigma'(\theta_n)$:

$$(16) \quad \sigma'(\theta) = \begin{cases} \sigma(\theta) & (\theta < 1/2) \\ \Psi & (\theta > 1/2). \end{cases}$$

Then (9) follows if we appropriately choose the words ω'_n to accurately encode some string of the initial orbit of 0^+ in I_n . Then the resulting word

$$(\sigma'_0 \circ \sigma'_1 \circ \cdots \circ \sigma'_{n-1})(\omega'_n)$$

will accurately represent the initial orbit of 0^+ , but it is no longer guaranteed that the length of this word increases! For example, if $\theta = [3, 2, 2, 2, \dots]$, then we will alternate between σ'_n being Ψ and a substitution which maps $C \rightarrow A$. Setting $\omega'_n = A$ for all those n for which $\theta_n < 1/2$ would therefore always map via this long string of compositions to

$$A \xrightarrow{\Psi} C \xrightarrow{\sigma} A \xrightarrow{\Psi} C \xrightarrow{\sigma} \cdots$$

Define

$$(17) \quad \omega'_n = \begin{cases} A^{k+1}B^{k-1}C & (a_1(\theta_n) = 2k) \\ A^{k+1}B^k & (a_1(\theta) = 2k + 1) \\ \Psi(\omega'_{n+1}) & (a_1(\theta) = 1). \end{cases}$$

The reader may verify that the word ω'_n does accurately encode some initial portion of the orbit of 0^+ depending on the parity of $a_1(\theta_n)$. Note that whenever Ψ is applied, it affects only the first letter of its input. From this it follows that if $\omega = (\omega)_0\nu$, then

$$(18) \quad (\sigma'_0 \circ \cdots \circ \sigma'_{n-1})(\omega) = (\sigma'_0 \circ \cdots \circ \sigma'_{n-1})((\omega)_0)(\sigma_0 \circ \cdots \circ \sigma_{n-1})(\nu).$$

As ω'_n always has length larger than one, our previous reasoning now guarantees that the length of Ω'_n diverges, establishing (9) and completing the proof.

Before moving on to the study of the growth rates of discrepancy sums, we present a few observations about this process.

Proposition 4.3. *Those θ for which $x(\theta) = 0 (= 0^+)$ are exactly the set*

$$(19) \quad H = \{\theta : a_{2i-1}(\theta) = 0 \bmod 2, i = 1, 2, \dots\}.$$

Proof. We leave the reader to verify that H is exactly the set of θ for which $g^n(\theta) < 1/2$ for every n . For those $\theta \in H$, then, we always have $I'_{n+1} = [0^+, \delta_n^-]$, where $\delta_n < 1$, and we never need apply the isomorphism $\varphi_n(x) = 1 - x$. That is,

$$0 \in (\varphi_0^{-1} \circ \cdots \circ \varphi_{n-1}^{-1})(I_n)$$

for all n : $0 = x(\theta)$.

On the other hand, if n is the first index such that $\theta_n > 1/2$, we must have $\varphi_n(x) = 1 - x$. As $\theta_{n+1} < 1/2$, however, it follows that within I_n , we have

$$\varphi_n^{-1} \circ \varphi_{n+1}^{-1}(I_{n+2}) = [(1 - \delta_{n+1})^+, 1^-],$$

from which it follows that

$$0 \notin (\varphi_0^{-1} \circ \dots \circ \varphi_{n+1}^{-1})(I_{n+2}). \quad \square$$

Proposition 4.4. *The sequence of substitutions σ_n is eventually periodic if and only if θ is a quadratic surd.*

Proof. Clearly the sequence σ_n is eventually periodic if and only if the orbit of θ under g is eventually periodic. From the definition (10) of g we have for all $i \geq 2$

$$(20) \quad a_i(\theta_{n+1}) = a_{i+k}(\theta_n) : \quad k = \begin{cases} 0 & (a_1(\theta_n) = 1 \bmod 2, \neq 1) \\ 1 & (a_1(\theta_n) = 1) \\ 2 & (a_1(\theta_n) = 0 \bmod 2) \end{cases}$$

So, if $a_i(\theta)$ are eventually periodic (Gauss' criteria for quadratic surds), we must have infinitely many n such that for all $i \geq 2$ we have for any j, k

$$a_i(\theta_{n_k}) = a_i(\theta_{n_j}).$$

Suppose that a period of $a_i(\theta)$ is given by the terms $\alpha_1, \dots, \alpha_N$, and assume without loss of generality that for $i \geq 2$

$$a_i(\theta_{n_k}) = \alpha_{i \bmod N}.$$

Then $a_1(\theta_{n_k})$ is either 1, α_1 , or $\alpha_1 + 1$. Since the collection n_k was infinite, one value must be taken twice, giving a period in the orbit $g(\theta)$.

On the other hand, assume that $\theta_j = \theta_{j+n_k}$ for $n = 0, 1, \dots$ and $k \neq 0$. From (20) it follows that $a_i(\theta)$ is eventually periodic. \square

Remark. The periods under g and γ need not be the same, nor is one necessarily longer than the other. For example, the golden mean has period one under γ but period two under g , while $\theta = [2, 1, 2, 1, \dots]$ has period two under γ and period one under g . Furthermore, the sequence σ_n is purely periodic if and only if $\theta_n = \theta_0$ for some $n \neq 0$, which is not the same as the partial quotients of θ being purely periodic. Consider for example $\theta = [3, 2, 2, 2, \dots]$, whose partial quotients are clearly not purely periodic, but satisfies $\theta_2 = \theta_0$.

5. THE ARITHMETIC OF OUR SUBSTITUTIONS

Let $\theta_0 < 1/2$, so that

$$f(x) = \begin{cases} +1 & (x \in A) \\ -1 & (x \in B \cup C). \end{cases}$$

For $\theta_0 > 1/2$ we could repeat all future arguments with a sign change. Given $\omega \in \mathcal{A}^n$, define (consistent with existing notation)

$$S(\omega) = \sum_{i=0}^{n-1} (\chi_A - \chi_{B \cup C}) \omega_i,$$

$$M(\omega) = \max \{S(\omega_0 \dots \omega_{j-1}) : j = 1, 2, \dots, n\},$$

$$m(\omega) = \min \{S(\omega_0 \dots \omega_{j-1}) : j = 1, 2, \dots, n\}.$$

Note that we *do not* include the empty word in determining $M(\omega)$, $m(\omega)$.

Proposition 5.1. *Suppose $|\omega| = n \neq 0$, $\omega \neq C$, $M(\omega) \geq 0$, ω does not have CC , CB or BA as factors, and σ is a substitution given by Table 2, depending on θ . If $a_1(\theta) = 0 \bmod 2$ and $a_3(\theta) \neq 1$, or if $a_1(\theta) = 1$, then:*

$$S(\sigma(\omega)) = S(\omega), \quad M(\sigma(\omega)) = M(\omega) + E(a_1), \quad m(\sigma(\omega)) = m(\omega).$$

On the other hand, if $a_1(\theta) = 0 \bmod 2$ and $a_3(\theta) = 1$, then

$$S(\sigma(\omega)) = -S(\omega), \quad M(\sigma(\omega)) = -m(\omega) + E(a_1), \quad m(\sigma(\omega)) = -M(\omega).$$

Finally, if $a_1(\theta) = 1 \bmod 2$, $\neq 1$, and either

- $(\omega)_{n-1} \neq C$, or
- $(\omega)_{n-1} = C$, but there is some $j \neq n$ such that $S((\omega)_0(\omega)_1 \dots (\omega)_{j-1}) = m(\omega)$,

then also

$$S(\sigma(\omega)) = -S(\omega), \quad M(\sigma(\omega)) = -m(\omega) + E(a_1), \quad m(\sigma(\omega)) = -M(\omega).$$

If $a_1(\theta) = 1 \bmod 2$, $(\omega)_{n-1} = C$ and $S((\omega)_0 \dots (\omega)_{j-1}) > m(\omega)$ for all $j \neq n$, then

$$S(\sigma(\omega)) = -S(\omega), \quad M(\sigma(\omega)) = -m(\omega) - 1 + E(a_1), \quad m(\sigma(\omega)) = -M(\omega).$$

Proof. The prohibition on CB , CC and BA being factors of ω are necessary for ω to encode the orbit of any point under rotation by any θ , so this condition is not prohibitive in our setting.

In all cases, the statements regarding the value $S(\sigma(\omega))$ follow from examining $S(\sigma(x))$ for each $x \in \mathcal{A}$; the reader may consult Table 2 to verify that $S(\sigma(x)) = \pm S(x)$ as described, and the statement then follows from the fact that σ is a homomorphism. We will turn our attention, then, to the statements regarding $m(\sigma(\omega))$ and $M(\sigma(\omega))$. All cases but the last are considered similarly with the possible sign-change outlined above in mind.

For example, suppose that $a_1 = 0 \bmod 2$ and $a_3 \neq 1$. Let $\omega = v\psi$, where v is the largest left factor of ω such that $S(v) = M(\omega) - 1$: note that as $M(\omega) \geq 0$ and the empty word was not considered in computation of $M(\omega)$, we have $(\psi)_0 = A$. As $S(\sigma(v)) = S(v) = M(\omega) - 1$ and $M(\sigma(A)) = E(a_1) + 1$, we know that

$$M(\sigma(\omega)) \geq M(\sigma(v)\psi) = M(\omega) + E(a_1).$$

Assume on the other hand that

$$\sigma(\omega) = \sigma(v)\nu\psi, \quad S(\sigma(v)\nu) > M(\omega) + E(a_1),$$

and v is of maximal length to allow such a decomposition. Note that $\nu \neq \emptyset$ as $S(\sigma(v)) = S(v) \leq M(\omega)$. As v is a proper factor, it is followed by a letter, and by maximality on the length of v , ν is a proper left factor of either $\sigma(A)$, $\sigma(B)$, or $\sigma(C)$, and $E(a_1) \neq 0$. If v is followed by A in ω ,

$$S(\sigma(v)) = S(v) \leq M - 1.$$

On the other hand, $S(\nu) \leq E(a_1) + 1 = M(\sigma(A))$, contradicting the value $S(\sigma(v)\nu)$. The possibility of v followed by B or C are similarly considered; the larger possible $S(\sigma(v)) = M(\omega)$ is countered by $S(\nu) \leq E(a_1)$ in these cases.

The ambiguity in the situation when $a_1(\theta) = 1 \bmod 2$, $\neq 1$ is due to the substitution $\sigma(A) = C$, which does not achieve an intermediate sum of $E(a_1)$ (as does $\sigma(B)$). On the assumption that there is some proper left factor ψ of ω such that

$S(\psi) = m(\omega)$, however, we know that the letter which follows ψ must be A ; similar computations to the above then apply. If the only left factor of ω which achieves a sum of $m(\omega)$ is in fact ω itself, then if the final letter of ω is B we again have no problem.

Assume, then, that $S(\omega) = m(\omega)$, there is no proper left factor with this sum, and ω ends with the letter C . As $M(\omega) \geq 0$ by assumption, there is a letter preceding this terminal C (that is, $\omega \neq C$). If this letter is A , then the left factor ψ such that $\omega = \psi AC$ has the minimal sum as its sum (even if it is empty), and the preceding reasoning applies. Therefore ω must be of the form ψBC (recall that CC is not a factor): considering $\sigma(B)$ following $S(\sigma(\psi)) = -m(\omega) - 2$ completes the proposition. \square

For convenience, denote

$$(21) \quad \sigma^{(n)} = \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_{n-1},$$

$$(22) \quad \sigma'^{(n)} = \sigma'_0 \circ \sigma'_1 \circ \cdots \circ \sigma'_{n-1}.$$

Recall (17) and define for $n \geq 1$

$$(23) \quad \Omega_n = \sigma^{(n)}(A), \quad \Omega'_n = \sigma'^{(n)}(\omega'(n)).$$

Define p_n to track the parity of how many $\theta_i > 1/2$:

$$(24) \quad p_n = \left(\sum_{i=1}^{n-1} \chi_{(1/2,1)}(\theta_i) \right) \bmod 2.$$

We now have all the tools necessary to precisely study the sequences $M_n(y)$ and $m_n(y)$ for $y \in \{x(\theta), 0\}$:

Proposition 5.2. *Assume that $\theta_0 < 1/2$. Then*

$$S(\Omega_n) = (-1)^{p_n}, \quad S(\Omega'_n) = 1$$

$$\begin{aligned} \left| M(\Omega_n) - \left(1 + \sum_{\substack{i \leq n-1 \\ p_i=0}} E(a_1(\theta_i)) \right) \right| &\leq 1, & M(\Omega'_n) &= 1 + \sum_{\substack{i \leq n \\ p_i=0}} E(a_1(\theta_i)), \\ \left| m(\Omega_n) - \left(1 - \sum_{\substack{i \leq n-1 \\ p_i=1}} E(a_1(\theta_i)) \right) \right| &\leq 1, & m(\Omega'_n) &= 1 - \sum_{\substack{i \leq n \\ p_i=1}} E(a_1(\theta_i)). \end{aligned}$$

Proof. The word Ω_n in (23) is formed by successive substitutions acting on the word A ; as such, it will always begin with A , so $M(\Omega_n) \geq 1$. We immediately see that all $S(\Omega_n) = \pm 1$ according to the parity of p_n by applying Proposition 5.1 in succession. The ambiguous case in Proposition 5.1 arose when ω was a word which had a nonnegative maximal sum (as do all Ω_n) and whose minimum sum is only achieved as its total sum, with C as a terminal factor. Furthermore, we would need θ_n to have first partial quotient odd and larger than one. For this to happen with the restriction that all $S(\Omega_n) = \pm 1$ requires that $S(\Omega_n) = -1$ (otherwise the minimal sum is achieved by the proper left factor A), and therefore $S(\Omega_{n-1}) = 1$. This scenario *also* require that $M(\Omega_{n-1}) = 1$ (otherwise $m(\Omega_n) < -1 \leq S(\Omega_n)$); so this situation can only occur in our scenario when $\Omega_{n-1} = A$: *this possible error of*

one may only appear once in the sequence of arithmetic computations from repeated application of Proposition 5.1.

We leave to the reader the verification that the parity of p_n exactly dictates whether substitutions will add to the maximal values or subtract from the minimal values; refer to Proposition 5.1 again.

Let us now consider Ω'_n . Note that $\sigma'_j = \Psi$ exactly when $\theta_j > 1/2$, exactly when σ_{j-1} has the property that $S(\sigma_{j-1}(\omega)) = -S(\omega)$. Clearly we have $S(\Psi(\omega)) = S(\omega) - 2$ provided ω begins with A . Also note that if $S(\omega) = 1$, then if $m(\omega) = 1$ we must have $\omega_0 = A$: it is never possible in our construction for ω to terminate with C , $S(\omega) = 1$, and $m(\Psi(\omega)) = S(\Psi(\omega))$ is the only time this value is reached.

Our choice of $\omega'(n)$ always begins with A and has $S(\omega'(n)) = 1$, and for those σ_n such that $S(\sigma_n(A)) = -1$, the reader may verify that

$$S(\sigma_n(\Psi(\omega))) = 2 - S(\omega)$$

by applying Proposition 5.1. While this change will change the sum of $+1$ to -1 , it is immediately followed by a substitution which reverses the sign of the sum: we maintain

$$S(\Omega'_n) = 1.$$

Furthermore, as $m(\omega'_n) = 1$ for all ω'_n , if we do apply Ψ (so $m(\Psi\omega) = -1$) followed by one of these sign-reversing substitutions σ , we see

$$M(\sigma(\Psi\omega)) \geq -m(\Psi\omega) + E(a_1) - 1 \geq 1 + E(a_1) - 1 \geq 0,$$

so we may always apply Proposition 5.1 without worrying about the possible error of one. \square

Corollary 5.3 ([2], Theorem 1, case $k = 2$). *We have $S_n(\theta) \geq 0$ for all $n \geq 0$ if and only if $x(\theta) = 0$.*

Proof. By viewing the ergodic sums as an additive cocycle, for all $n > 0$ we have $S_n(\theta) = S_{n+1}(0) - 1$, so we have by Proposition 5.2:

$$S_{|\Omega'_n|-1}(\theta) = 0, \quad M_{|\Omega'_n|-1}(\theta) = \sum_{\substack{i \leq n \\ p_i = 0}} E(a_1(\theta_i)), \quad m_{|\Omega'_n|-1}(\theta) = - \sum_{\substack{i \leq n \\ p_i = 1}} E(a_1(\theta_i)).$$

So $S_n(\theta) \geq 0$ for all n if and only if $p_i = 0 \bmod 2$ for all i such that $\theta_i < 1/2$, which is equivalent to $p_i = 0 \bmod 2$ for all i . A direct inductive argument shows that $p_i = 0$ for all i if and only if $a_{2i-1}(\theta) = 0 \bmod 2$ by considering the action of $g(10)$, which corresponds by Proposition 4.3 to $x(\theta) = 0$. \square

Remark. Using that σ are all homomorphisms, a more constructive version of (7) is

$$\omega_0 \sigma^{(1)}(\omega_1) \sigma^{(2)}(\omega_2) \cdots \sigma^{(n)}(\omega_n) \cdots,$$

which allows a more direct way of computing the word through successive computation of the words ω_n (given the starting point x).

Lemma 5.4. *We always have*

$$\left| \sigma^{(n)}(A) \right| = \left| \sigma^{(n)}(B) \right|,$$

and if we define the matrices $M_i = M(\theta_i)$ according to Table 3, then

$$M_{n-1} M_{n-2} \cdots M_1 M_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} |\sigma^{(n)}(A)| \\ |\sigma^{(n)}(C)| \end{bmatrix}.$$

Proof. The first claim follows directly from the following observation: for all substitutions σ , the words $\sigma(A)$ and $\sigma(B)$ are always of the same length and always contain the same number of letters drawn from $\{A, B\}$. That is, within

$$(\varphi_{n-1}^{-1} \circ \cdots \circ \varphi_0^{-1})(A \cup B) \subset \tilde{I}_n$$

the return time under R_{θ_0} to \tilde{I}_n is constant, and similarly on the pullback of C . One need only count the number of C and $\{A, B\}$ within $\sigma_n(C)$ and $\sigma_n(\{A, B\})$ to construct the relevant matrices. \square

Case	$M(\theta)$	
$a_1(\theta) = 0 \bmod 2, a_3(\theta) \neq 1$	$(a_1 - 1)a_2 + 1$	a_2
	$(a_1 - 1)a_2 + a_1$	$a_2 + 1$
$a_1(\theta) = 0 \bmod 2, a_3(\theta) = 1$	$(a_1 - 1)a_2 + a_1$	$a_2 + 1$
	$(a_1 - 1)a_2 + 1$	a_2
$a_1(\theta) = 1 \bmod 2, \neq 1$	$a_1 - 1$	1
	1	0
$a_1(\theta) = 1$	1	0
	0	1

TABLE 3. The matrices $M(\theta)$ used to determine return times in the induced systems.

Lemma 5.5.

$$|\Omega_n| \leq |\Omega'_n| \leq |\Omega_{n+1}|.$$

Proof. The lower inequality is direct in light of (18), recalling that $(\omega'_n)_1 = A$. The upper bound follows from Lemma 5.4, noting that while ω'_n may or may not be a left factor of $\sigma_n(A)$, it does contain the same number of $\{A, B\}$ versus C as a proper left factor of $\sigma_n(A)$. Furthermore, the only substitutions for which $|\sigma(C)| > |\sigma(A)|$ are those corresponding to $a_1 = 0 \bmod 2, a_3 \neq 0$; such substitutions are *not* followed by Ψ . That is,

$$|\sigma'^{(n)}(A)| \leq |\Omega_n|,$$

completing the proof of the upper bound. \square

Example 5.6. Let $\theta = \sqrt{2} \bmod 1 = [2, 2, 2, \dots]$. Then as θ is a quadratic irrational, the sequence of substitutions σ_i is eventually periodic by Proposition 4.4. As $g(\theta) = \theta$, the sequence of substitutions is periodic with period one, given by

$$\sigma : \begin{cases} A \rightarrow AACAC \\ B \rightarrow ABCAC \\ C \rightarrow ABCACAC \end{cases}$$

The point $x(\theta) = 0$ by Proposition 4.3, so applying Theorem 1.1, the orbit of zero is given by the sequence

$$\lim_{n \rightarrow \infty} \sigma^n(A) = AACACAACACABCACACAACACABCACAC \dots$$

The self-similar structure of the sequence of ergodic sums $S_n(0)$ is not exact (as $\sigma(B) \neq \sigma(C)$), but nonetheless highly regular. This regularity was noticed by D. Hensley in [4, Figure 3.4]. We give several plots of $S_n(0)$ for different values of n

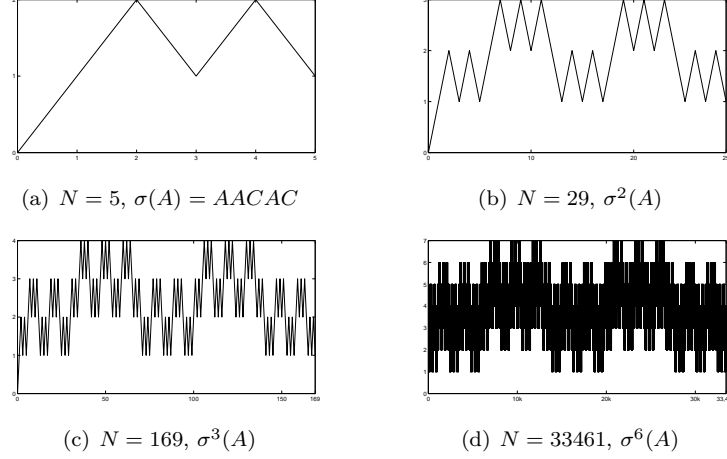


FIGURE 2. Plots of $S_i(0)$ for different ranges of $0 \leq i \leq N$, where $\theta = \sqrt{2} - 1$.

in Figure 2. This same self-similarity for developing the orbit of $x(\theta)$ will be seen for any quadratic irrational θ in light of Proposition 4.4.

For quadratic irrational $\theta \notin H$, computation of the point $x(\theta)$ is not too difficult:

Example 5.7. Let $\theta = [1, 1, \dots]$ be the golden mean. Recall that S^1 will be partitioned such that $A = [(1/2)^+, 1^-]$ as $\theta > 1/2$. As $g^2(\theta) = \theta$, and $a_1 = 1$ corresponds to the identity substitution, the only non-identity substitution generated is

$$\sigma : \begin{cases} A \rightarrow ABCAC \\ B \rightarrow AACAC \\ C \rightarrow AAC \end{cases}$$

So, the orbit of $x(\theta)$ is given by

$$\lim_{n \rightarrow \infty} \sigma^n(A) = ABCACAACACAACABCACAAC \dots,$$

while the orbit of 0 is given by

$$\Psi(\sigma(\dots \Psi(AAC))) = CACABCACAACABCACAACAC \dots$$

To compute the point $x(\theta)$, we need to determine the intervals \tilde{I}_n . For those $\theta_n = [2, 1, 1, \dots]$ we have

$$\delta_n = 1 - 2\theta_n = 1 - 2(1 - \theta) = 2\theta - 1.$$

Denote this quantity by δ for convenience. For this particular θ we do not ever have two consecutive $\theta_n < 1/2$, so the intervals $I'_{n+1} \subset I_n$ strictly alternate between $[0^+, \delta^-]$ and $[(1 - \delta)^+, 1^-]$ (for those $n = 0 \bmod 2$; for odd n we have $\theta_n > 1/2$ and $I'_{n+1} = I_n$). So the sequence of preimages \tilde{I}_n (recall again (14)) is given by

$$[0^+, 1^-], \quad [(1 - \delta)^+, 1^-], \quad [(1 - \delta)^+, (1 - \delta + \delta^2)^-], \dots$$

whose intersection is given by the geometric series

$$x(\theta) = \sum_{i=0}^{\infty} (-1)^i \delta^i = \frac{1}{1 + (2\theta - 1)} = \frac{1}{2\theta}.$$

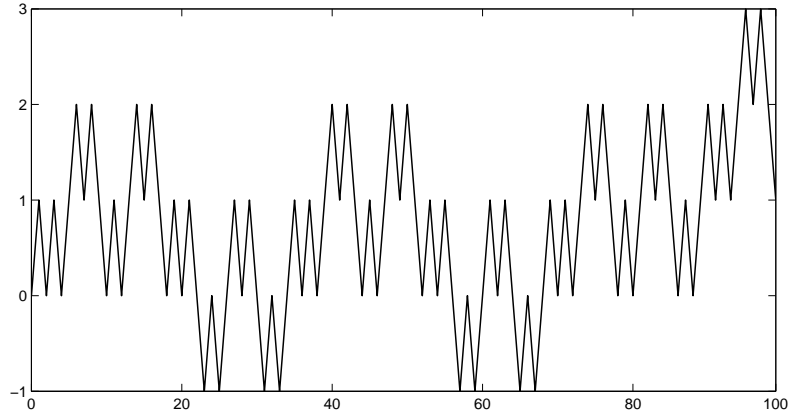
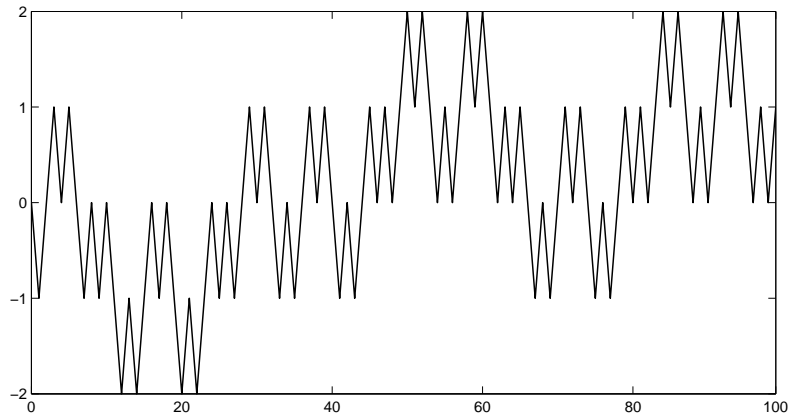

 (a) $x = 0$, with orbit $CACABCACAAC \dots$

 (b) $x = x(\theta) = 1/(2\theta)$, with orbit $ABCACAACAC \dots$

FIGURE 3. Plots of $S_i(x)$ for $0 \leq i \leq 100$, where θ is the golden mean for the two given values of x . Note that as $\theta > 1/2$, we have $A \rightarrow -1$, $B, C \rightarrow +1$.

See Figure 3 for both of these orbits.

One particularly striking corollary of Proposition 5.2 is the following, which does not seem to be apparent from any other technique:

Corollary 5.8. *If θ is a quadratic irrational, then*

$$\lim_{n \rightarrow \infty} \frac{M_n(0)}{|m_n(0)|} \in \mathbb{Q}^*,$$

where $\mathbb{Q}^* = \mathbb{Q} \cup \{\infty\}$, and $p/0 = \infty$ for any positive integer p . If $\theta_n = \theta_{n+k}$ is a minimal period under the orbit of g and $p_{n+k} = p_n + 1$, then the ratio tends to one.

Furthermore, for any nonnegative $p/q \in \mathbb{Q}^*$, there is a quadratic irrational θ such that the above ratio has limit p/q .

Proof. We have already shown that $g^n(\theta)$ is eventually periodic for such θ in Proposition 4.4. It follows from Proposition 5.2 that $M_n(0)$ and $m_n(0)$ see a periodic sequence of adjustments by bounded integer amounts, which must therefore have rational limit. If one period reflects a change in the parity of p , it will always be followed by the mirrored changes in M_n, m_n , producing a limit of one.

To produce quadratic irrationals with the desired limit, if $q = 0$ then $\theta \in H$ will suffice ($m_n(0) \equiv 1$, and $M_n(0)$ must therefore diverge), and for $p = 0$ any θ such that $a_1(\theta) = 1$ and $g(\theta) \in H$ will suffice (here $M_n(0) \equiv 1$). For p/q with neither zero, just set

$$\theta = [2p, 1, 1, 2q - 1, 1, 1, 2p - 1, 1, 1, 2q - 1, 1, 1, \dots],$$

and verify that we will first add p to $M_n(0)$, then subtract q from $m_n(0)$, etc. \square

6. PROOF OF THEOREM 1.2

Let c_n and d_n be divergent monotone sequences in $o(n)$ with bounded differences $\Delta c_n, \Delta d_n$; we will construct a dense set of θ such that

$$\limsup_{n \rightarrow \infty} \frac{M_n(0)}{c_n} = \limsup_{n \rightarrow \infty} \frac{|m_n(0)|}{d_n} = 1.$$

Any irrational θ is completely determined by its sequence of partial quotients, which is equivalent to its orbit under g , and its orbit under g is completely determined by the sequence of values

$$a_1(\theta_i) \quad (a_1 = 1 \bmod 2), \quad a_1(\theta_i), a_2(\theta_i) \quad (a_1 = 0 \bmod 2).$$

Suppose, then, that the first finitely many partial quotients of θ are prescribed, such that the first n values of θ_i are fixed. Without loss of generality, insert an additional single term if necessary so that $p_n = 0$ (recall (24)). We are now completely free to choose k to construct ω'_n (refer to (17)). If we denote

$$M(\Omega'_n) = M, \quad m(\Omega'_n) = m, \quad |\Omega'_n| = L_n,$$

it follows from Proposition 5.2 that once we choose k , we will have

$$M(\Omega'_{n+1}) = M + k, \quad m(\Omega'_{n+1}) = m.$$

Denote by $L_{n+1}(k) = |\Omega'_{n+1}|$ as a function of k .

Assume first that $M < c_{L_n}$, so we wish to increase the maximal sum compared to the sequence c_n . Then let $a_1(\theta_n)$ be odd, so

$$\omega'(n+1) = A^{k+1}B^k.$$

From (18) and the previous observation that $|\sigma^{(n)}(A)| = |\sigma^{(n)}(B)|$, it follows that

$$L_{n+1}(k) = |\tilde{\omega}| + 2k|\sigma^{(n+1)}(A)|,$$

where

$$\tilde{\omega} = \sigma'^{(n+1)}(A).$$

Consider, then, the proper left factors A^i of $\omega'(n+1)$ for $i = 1, 2, \dots, k+1$. Applying Proposition 5.1, the new maximal sum $M + k$ is achieved at a time N , where

$$|\tilde{\omega}| + (k-1)|\sigma^{(n)}(A)| \leq N \leq |\tilde{\omega}| + k|\sigma^{(n)}(A)|.$$

As $c_n \in o(n)$, we may choose $k \geq 1$ to be minimal such that

$$\frac{M + k}{c(|\tilde{\omega}| + k|\sigma^{(n)}(A)|)} \geq 1.$$

If, however, we had $M \geq c_{L_n}$, then we would wish to not greatly increase M compared to c_n . In this case, let $\theta_n = [2, k, 1, \dots]$, and pass directly to considering the word

$$\sigma'^{(n+1)}(C) = \sigma'^{(n)}(A^{k+1}B^{k-1}C),$$

as C is always a left factor of $\omega'_{n+1} = \Psi(\omega'_{n+2})$ in this case. Then the maximal sum reached for this word is $M + 1$, but its length is (similarly to before)

$$L_{n+1}(k) = |\tilde{\omega}| + 2k|\sigma^{(n)}(A)|.$$

We are now in the position of being able to increase the length of the word *without* increasing the maximal sum of $M + 1$, so as c_n is divergent, choose $k \geq 1$ minimal such that

$$\frac{M + 1}{c(|\tilde{\omega}| + k|\sigma^{(n)}(A)|)} \leq 1.$$

After applying g twice (to skip past the next $\theta_k > 1/2$), then, we find ourselves able to manipulate the growth of the minimal sums $m(n)$. Continuing in this fashion, then, we construct a dense set of θ (as the initial string of partial quotients was arbitrary). That the limsups are actually one follows from the minimal choice of k and that $\Delta c_n, \Delta d_n$ are bounded.

To prove the analogous statements where one of M_n, m_n is desired to remain bounded, one need only repeat the same arguments using $\theta_n \in H$ (recall (19)) so that the value p_n is eventually constant.

The statement of Theorem 1.2 applies as well to $M_n(x(\theta))$ and $m_n(x(\theta))$; the proof is simpler, in fact, as the map Ψ is not a concern, and the possible error of one from Proposition 5.2 is not an asymptotic concern. This process is highly amenable to diagonalization techniques. For example:

Corollary 6.1. *Given a countable collection of sequences $c_n^{(i)}$ and $d_n^{(i)}$, all of which are divergent and in $o(n)$, such that*

$$c_n^{(1)} \leq c_n^{(2)} \leq \dots, \quad d_n^{(1)} \geq d_n^{(2)} \geq \dots,$$

there is a dense set of θ for which

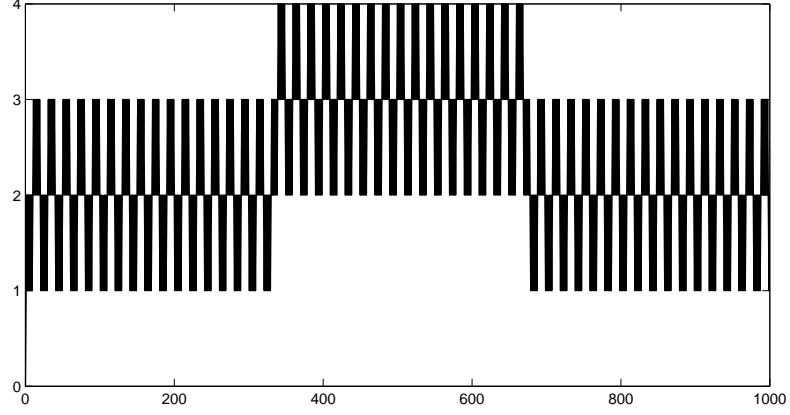
$$c_n^{(i)} \in o(M_n(0)), \quad |m_n(0)| \in o(d_n^{(i)})$$

for all i .

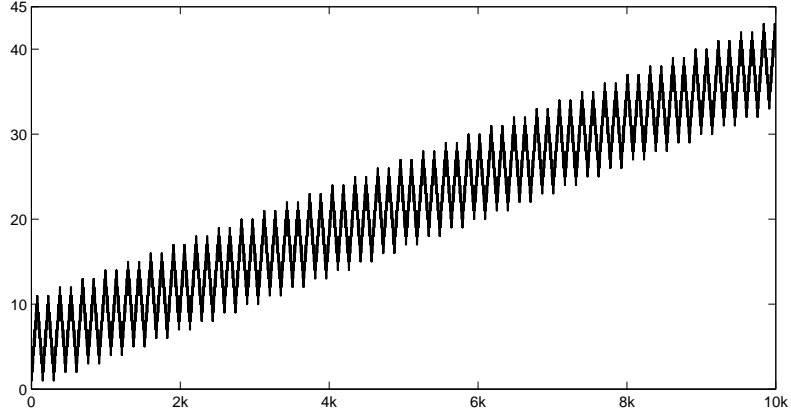
Proof. Apply Theorem 1.2 after using a diagonalization process to construct c_n, d_n , both monotone, divergent, and in $o(n)$ such

$$c_n^{(i)} \in o(c_n), \quad d_n \in o(d_n^{(i)}). \quad \square$$

Many permutations of the above corollary are possible. For example, we may construct a dense set of θ such that the discrepancy sums grow in both directions faster than any $n^{1-\epsilon}$ (but necessarily in $o(n)$, of course!), or such that the discrepancy sums are bounded below, but $M_n(0)$ grows slower than all iterated logarithms (but necessarily divergent, of course!), etc. See Figure 4 for an example where



(a) θ exhibiting very slow growth of $M_n(0)$; this portion of the graph will repeat 2^{16} times with no additional growth.



(b) $\gamma(\theta)$ exhibiting very fast growth of $M_n(0)$; this sawtooth pattern will continue to climb by repeating itself $E(2^{16})/2$ times.

FIGURE 4. Two different extreme growth rates for θ and $\gamma(\theta)$.

for both θ and $\gamma(\theta)$ we have $m_n \geq 1$, but $M_n(\theta) \notin o(n^{1-\epsilon})$ for any $\epsilon > 0$ while $M_n(\gamma(\theta)) \in o(\log^{(i)} n)$ for all i . In Figure 4 we set

$$\theta = [2, 2^2, 2, 2^{2^2}, 2, 2^{2^{2^2}}, 2, \dots].$$

Using diagonalization techniques one may similarly find a dense set of θ such that

$$\limsup_{i \rightarrow \infty} \frac{M_{n_i(j)}(0)}{c_{n_i(j)}^{(j)}} = 1$$

for an arbitrary collection of divergent sequences $c_n^{(j)}$ in $o(n)$ for different subsequences $n_i(j) \rightarrow \infty$ depending on j , and similarly for the $|m_n(0)|$ and a collection of sequences $d_n^{(j)}$.

Truly, beyond the constraints of (2), any asymptotic behavior desired is possible.

7. PROOF OF THEOREM 1.3

Suppose that

$$(25) \quad \liminf_{n \rightarrow \infty} \frac{M_n(0)}{|m_n(0)|} = r_1, \quad \limsup_{n \rightarrow \infty} \frac{M_n(0)}{|m_n(0)|} = r_2.$$

That the set of accumulation points of the sequence is the entire closed interval $[r_1, r_2]$ is direct and is left to the reader. Let an arbitrary finite string of partial quotients a_1, \dots, a_N be given which determine θ_i for $i = 0, 1, \dots, n-1$, and for convenience again assume without loss of generality that $p_n = 0$.

Now let c_n and d_n be arbitrary integer-valued strictly increasing sequences such that Δc_n and Δd_n are in $O(1)$ and

$$\liminf_{n \rightarrow \infty} \frac{c_n}{d_n} = \rho_1, \quad \limsup_{n \rightarrow \infty} \frac{c_n}{d_n} = \rho_2.$$

Furthermore, assume that $c_1 > M(\Omega'_n) = M$ and $d_1 > |m(\Omega'_n)| = m$.

Continue the continued fraction expansion of θ in the following way:

$$\theta_n = [2(c_1 - M) + 1, 2(d_1 - m), 2(c_2 - c_1), 2(d_2 - d_1), \dots].$$

Then Ω'_n will see the sequence of $M(\Omega'_{n+2k}) = c_k$ and $m(\Omega'_{n+2k}) = -d_k$; the bounded differences Δc_n and Δd_n ensure that the limiting behavior is the same as the limiting behavior along the subsequence of times $|\Omega'_n|$.

Example 7.1. Suppose that $\theta = [1, 2, 3, 4, \dots]$. Then we begin computing the sequence of values $M_n(0)$ and $|m_n(0)|$ according to Proposition 5.2:

$$(26) \quad \begin{array}{|c|c|c|c|} \hline \theta_0 = [1, 2, 3, 4, \dots] & p = 0 & E(a_1) = 0 & (M, |m|) = (1, 1) \\ \theta_1 = [3, 3, 4, 5, \dots] & p = 1 & E(a_1) = 1 & (M, |m|) = (1, 0) \\ \theta_2 = [1, 3, 4, 5, \dots] & p = 1 & E(a_1) = 0 & (M, |m|) = (1, 0) \\ \theta_3 = [4, 4, 5, 6, \dots] & p = 0 & E(a_1) = 2 & (M, |m|) = (3, 0) \\ \theta_4 = [5, 6, 7, 8, \dots] & p = 0 & E(a_1) = 2 & (M, |m|) = (5, 0) \\ \hline \theta_5 = [1, 6, 7, 8, \dots] & p = 0 & E(a_1) = 0 & (M, |m|) = (5, 0) \\ \theta_6 = [7, 7, 8, 9, \dots] & p = 1 & E(a_1) = 3 & (M, |m|) = (5, 3) \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline \end{array}$$

The pattern is seen to continue in groups of five terms. Over the terms θ_{5k} through θ_{5k+4} , we will subtract $2k+1$ from m while adding $2(2k+2)$ to M . We therefore have $\rho_1 = \rho_2 = 2$, or

$$\lim_{n \rightarrow \infty} \frac{M_n(0)}{|m_n(0)|} = 2.$$

See Figure 5 for this θ .

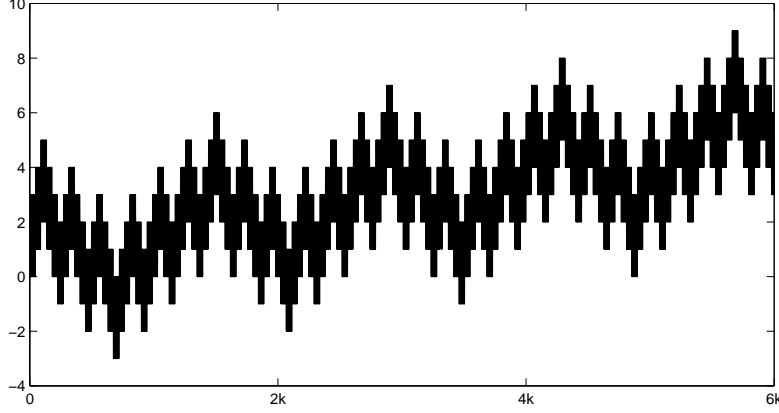


FIGURE 5. A specific θ for which $M_n(0)/|m_n(0)|$ has limit two; refer to (26) and note the changes to M , m .

8. PROOF OF THEOREM 1.4

Lemma 8.1. *Suppose that $f(x)$ is a step function on S^1 with $k < \infty$ discontinuities, and denote $V(f)$ the variation of f . Define $S_n(x)$, $M_n(x)$ and $m_n(x)$ as before. As we have not restricted f to be integer-valued, define*

$$\rho_N(x) = (M_N - m_N)(x).$$

Let n be such that $q_n \leq N < q_{n+1}$. Then for any $x, y \in S^1$:

$$\rho_N(y) \leq \rho_{q_{n+2}}(x) + a_{n+1}V(f).$$

Proof. Consider the set $\{x + i\theta\}$ for $i = 0, 1, \dots, q_n - 1$. Choose $0 \leq j < q_n$ such that $x + j\theta$ is closest to y . Then the distance between $x + j\theta$ and y is no larger than q_n^{-1} . For each discontinuity d_i there are therefore at most a_{n+1} preimages of d_i within this interval for time $L = 0, 1, \dots, q_{n+1} - 1$. It follows that $f(x + (j + i)\theta) = f(y + i\theta)$ for all but at most $k \cdot a_{n+1}$ of $i = 0, 1, \dots, N < q_{n+1}$. As $j + i$ is less than $q_n + q_{n+1} \leq q_{n+2}$, the lemma follows. \square

Assume that $a_i(\theta) \leq M$ for all i . Then (continuing with existing notation) we see that for some $C > 1$ independent of θ

$$(27) \quad C^{\frac{n-1}{2}} \leq |\Omega'_n| \leq (M+1)^{2n+2}.$$

The lower bound is due to the exponential decay in the length of the interval \tilde{I}_n (any $C < 2$ eventually suffices, as \tilde{I}_{n+1} is less than half as large as \tilde{I}_n at least half the time, with the $n-1$ accounting for the possibility that $I'_1 = I_0$, or $\theta_0 > 1/2$). The upper bound follows from Lemma 5.4, Lemma 5.5, and the bound $a_i(\theta) \leq M$ while at the same time,

$$(28) \quad \frac{n-1}{2} \leq \rho_{|\Omega'_n|}(0) \leq \frac{nM}{2};$$

the lower inequality is due to the fact that at most half of the words $\Omega'_n = \Omega'_{n+1}$ (corresponding to those $\theta_n > 1/2$) and for the rest, $\rho(\Omega_{n+1}) \geq \rho(\Omega_n) + 1$, as

$E(a_1) \geq 1$ for these $\theta_n < 1/2$. The upper bound follows as $E(a_i(\theta)) \leq M/2$ for all i .

Now, for any N let k be chosen such that

$$|\Omega_k| \leq N \leq |\Omega_{k+1}|.$$

From (27):

$$kC_1 \leq \log |\Omega'_k| \leq \log(N) \leq \log |\Omega'_{k+1}| \leq kC_2,$$

for two constants C_1 and C_2 which do not depend on k . From (28):

$$\frac{(k+1)M}{2} \geq \rho_{|\Omega'_{k+1}|}(0) \geq \rho_N(0) \geq \rho_{|\Omega'_k|}(0) \geq \frac{k-1}{2},$$

so $\rho_n(0) \sim \log(n)$. The full theorem now follows from Lemma 8.1.

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